Journal of Algebra **221,** 570–578 (1999)
Article ID jabr.1999.7996, available online at http://www.idealibrary.com on IDEAL®

A Property Equivalent to n-Permutability for Infinite Groups

Alireza Abdollahi* and Aliakbar Mohammadi Hassanabadi†

and

Bijan Taeri‡

Department of Mathematics, University of Technology of Isfahan, Isfahan, Iran

Communicated by Gernot Stroth

Received September 7, 1998

Let n be an integer greater than 1. A group G is said to be n-permutable whenever for every n-tuple $(x_1, \ldots, x_n)$ of elements of G there exists a non-identity permutation σ of $\{1, \ldots, n\}$ such that $x_1 \cdots x_n = x_{\sigma(1)} \cdots x_{\sigma(n)}$. In this paper we prove that an infinite group G is n-permutable if and only if for every n infinite subsets $X_1, \ldots, X_n$ of G there exists a non-identity permutation σ on $\{1, \ldots, n\}$ such that $X_1 \cdots X_n \cap X_{\sigma(1)} \cdots X_{\sigma(n)} \neq \varnothing$. © 1999 Academic Press

1. INTRODUCTION

Permutable groups have been studied by various people (for example, see [1–3, 5, 6]). Let n be an integer greater than 1. Recall that a group G is called n-permutable whenever for every n-tuple $(x_1, \ldots, x_n)$ of elements of G there exists a non-identity permutation σ of $\{1, \ldots, n\}$ such that $x_1 \cdots x_n = x_{\sigma(1)} \cdots x_{\sigma(n)}$. Also a group is said to be permutable if it is n-permutable for some integer $n > 1$. The main result for groups in this class was obtained by Curzio *et al.* in [3], where it was shown that such

* E-mail: abdolahi@math.ui.ac.ir.
† E-mail: aamohaha@math.ui.ac.ir.
‡ E-mail: b.taeri@cc.iut.ac.ir.

0021-8693/99 $30.00

groups are finite-by-abelian-by-finite. Let $n > 1$ and m be positive integers. Let S_n denote the group of all permutations on the set $\{1, \ldots, n\}$. A natural extension of permutable groups, namely (m, n)-permutable groups, groups in which $X_1 \cdots X_n \subseteq \bigcup_{\sigma \in S_n \setminus 1} X_{\sigma(1)} \cdots X_{\sigma(n)}$ for all subsets X_i of G where $|X_i| = m$ for all $i = 1, \ldots, n$, was introduced by Mohammadi Hassanabadi and Rhemtulla in [9]. It was proved there that such a group either is n-permutable or is finite of order bounded by a function of m and n. In [8] Mohammadi Hassanabadi investigated another extension of (m, n)-permutable groups as follows. For positive integers $n > 1$ and m a group G is called restricted (m, n)-permutable if $X_1 \cdots X_n \cap \bigcup_{\sigma \in S_n \setminus 1} X_{\sigma(1)} \cdots X_{\sigma(n)} \neq \varnothing$ for all subsets X_i of G where $|X_i| = m$ for all $i = 1, \ldots, n$. It was proved there that such a group is finite-by-abelian-by-finite. In [4] Longobardi *et al.* called a group G a P_n^*-group (n an integer greater than 1) if for every sequence $X_1, \ldots, X_n$ of infinite subsets of G there exist x_i in X_i such that $x_1 \cdots x_n = x_{\sigma(1)} \cdots x_{\sigma(n)}$ for some non-trivial permutation σ in S_n. They proved that every infinite P_n^*-group is an n-permutable group. Here we deal with another extension of infinite restricted (m, n)-permutable and P_n^*-groups.

Let n be an integer greater than 1. We call a group G a restricted (∞, n)-permutable group if $X_1 \cdots X_n \cap \bigcup_{\sigma \in S_n \setminus 1} X_{\sigma(1)} \cdots X_{\sigma(n)} \neq \varnothing$ for all infinite subsets $X_1, \ldots, X_n$ of G.

Our main result is the following, which sharpens and generalizes that of [8] and also generalizes the result of [4] concerning P_n^*.

THEOREM. *Every infinite restricted (∞, n)-permutable group is n-permutable.*

2. PROOFS

To prove the theorem, we need the following results.

LEMMA 2.1. *Let G be an infinite residually finite group which is a restricted (∞, n)-permutable group. Then G is an n-permutable group.*

Proof. Let $x_1, \ldots, x_n$ be arbitrary elements of G and

$$S = \left\{ x_1 \cdots x_n (x_{\sigma(1)} \cdots x_{\sigma(n)})^{-1} \,\middle|\, \sigma \in S_n \setminus 1 \right\}.$$

Suppose, for a contradiction, that $1 \notin S$. Since G is residually finite and S is finite, there exists a normal subgroup N of finite index in G such that $S \cap N = \varnothing$. Now considering infinite subsets $Nx_1, \ldots, Nx_n$, there exists $\sigma \in S_n \setminus 1$ such that $Nx_1 \cdots Nx_n \cap Nx_{\sigma(1)} \cdots Nx_{\sigma(n)} \neq \varnothing$ and so $x_1 \cdots x_n (x_{\sigma(1)} \cdots x_{\sigma(n)})^{-1} \in N$, which is a contradiction. ∎

LEMMA 2.2. *Let $G = Dr_{i\in I}G_i$ be an infinite direct product of non-abelian subgroups. Then G is not a restricted (∞, n)-permutable group for all integers $n > 1$.*

Proof. Suppose, for a contradiction, that G is a restricted (∞, n)-permutable group for some integer $n > 1$. We show that G is an n-permutable group, which contradicts Corollary 2.9 of [1]. Let $x_1, \ldots, x_n \in G$ and put $S = \{x_1 \cdots x_n(x_{\sigma(1)} \cdots x_{\sigma(n)})^{-1} \mid \sigma \in S_n\backslash 1\}$. Let k be any integer greater than $|S|$. Since G is an infinite direct product of normal subgroups, there exist k infinite normal subgroups $N_1, \ldots, N_k$ of G such that $N_i \cap N_j = 1$ for all distinct $i, j \in \{1, \ldots, k\}$. Let $l \in \{1, \ldots, k\}$ and consider infinite subsets $N_l x_1, \ldots, N_l x_n$. By the hypothesis, there exists $\sigma_l \in S_n\backslash 1$ such that $x_1 \cdots x_n(x_{\sigma_l(1)} \cdots x_{\sigma_l(n)})^{-1} \in N_l$. Therefore there exist two distinct $i, j \in \{1, \ldots, k\}$ and an element $s \in S$ such that $s \in N_i \cap N_j = 1$ and so G is an n-permutable group. ∎

We denote by A^{-1} the set $\{a^{-1} \mid a \in A\}$ for any non-empty subset A of a group. Let a and g be arbitrary elements of a group G. We define $S(a, g) := \{x \in G \mid a^x = g\}$ which is either an empty set or a right coset of the centralizer of a in G.

A key result required in the proof of the theorem is the following:

LEMMA 2.3. *Let G be an infinite restricted (∞, n)-permutable group. Then the FC-centre of G is non-trivial.*

Proof. Suppose, for a contradiction, that the FC-centre of G is trivial. We construct n infinite subsets $X_1, \ldots, X_n$ of G such that

$$X_1 \cdots X_n \cap X_{\sigma(1)} \cdots X_{\sigma(n)} = \varnothing$$

for all non-identity permutations σ in S_n. For this, for each $m \in \mathbb{N}$ we construct n subsets $X_{i,m} = \{a_{i,1}, \ldots, a_{i,m}\}$ of G $(i = 1, \ldots, n)$, such that

$$X_{1,m} \cdots X_{n,m} \cap X_{\sigma(1),m} \cdots X_{\sigma(n),m} = \varnothing \tag{\#}$$

for all non-identity permutations σ in S_n. We argue by induction on m. Let $m = 1$. By Lemma 2.1 in [3], G is not an n-permutable group and so there exist $a_{1,1}, \ldots, a_{n,1} \in G$ such that $a_{1,1} \cdots a_{n,1} \neq a_{\sigma(1),1} \cdots a_{\sigma(n),1}$ for all $\sigma \in S_n\backslash 1$. Now suppose that we have already defined subsets $X_{i,m} = \{a_{i,1}, \ldots, a_{i,m}\}$ of G $(i = 1, \ldots, n)$ satisfying $(\#)$ for all $\sigma \in S_n\backslash 1$.

Suppose that we have already defined $a_{i,m+1}$ and so $X_{i,m+1}$ for $i = 1, \ldots, r$ such that for all $\sigma \in S_n\backslash 1$

$$X_{1,m+1}X_{2,m+1} \cdots X_{r,m+1}X_{r+1,m} \cdots X_{n,m} \cap X_{\sigma(1),j_1} \cdots X_{\sigma(n),j_n} = \varnothing$$

where $j_t = m+1$ whenever $\sigma(t) \in \{1, \ldots, r\}$ and otherwise $j_t = m$. Let T_{r+1} be the union of all the following sets where σ varies over $S_n \backslash 1$,

$$X^{-1}_{\sigma(i-1), s_{i-1}} \cdots X^{-1}_{\sigma(1), s_1} X_{1, m+1} \cdots X_{r, m+1} X_{r+1, m} \cdots X_{n, m} X^{-1}_{\sigma(n), s_n} \cdots X^{-1}_{\sigma(i+1), s_{i+1}}$$

where $s_l = m+1$ whenever $\sigma(l) \in \{1, \ldots, r\}$ and otherwise $s_l = m$; also i varies over $\{1, \ldots, n\}$ and if $i = 1$ or $i = n$ then we define respectively $X^{-1}_{\sigma(i-1), s_{i-1}} \cdots X^{-1}_{\sigma(1), s_1} = \{1\}$ or $X^{-1}_{\sigma(n), s_n} \cdots X^{-1}_{\sigma(i+1), s_{i+1}} = \{1\}$. Also put

$$U_{r+1} = X^{-1}_{r, m+1} \cdots X^{-1}_{1, m+1} \left(\bigcup_{1 \neq \sigma \in S_n} X_{\sigma(1), j_1} \cdots X_{\sigma(n), j_n} \right) X^{-1}_{n, m} \cdots X^{-1}_{r+2, m} \cup T_{r+1}$$

where $j_l = m+1$ whenever $\sigma(l) \in \{1, \ldots, r\}$ and otherwise $j_l = m$. Now we prove that there exists an element $a_{r+1, m+1} \in G \backslash U_{r+1}$ such that if $X_{r+1, m+1} = \{a_{r+1, 1}, \ldots, a_{r+1, m+1}\}$ then for all $\sigma \in S_n \backslash 1$

$$X_{1, m+1} \cdots X_{r+1, m+1} X_{r+2, m} \cdots X_{n, m} \cap X_{\sigma(1), j_1} \cdots X_{\sigma(n), j_n} = \varnothing$$

where $j_l = m+1$ whenever $\sigma(l) \in \{1, \ldots, r+1\}$ and otherwise $j_l = m$. Suppose not. Therefore $a_{1, i_1} a_{2, i_2} \cdots a_{n, i_n} = a_{\sigma(1), j_1} a_{\sigma(2), j_2} \cdots a_{\sigma(n), j_n}$ for some $1 \le i_1, \ldots, i_{r+1} \le m+1$, $1 \le i_{r+2}, \ldots, i_n \le m$, $1 \le j_1, \ldots, j_n \le m$, and $1 \le j_s \le m+1$ whenever $\sigma(s) \in \{1, \ldots, r+1\}$. Suppose that $\sigma(t) = r+1$. If $i_{r+1} \neq m+1$ or $j_t \neq m+1$ then we get contradiction with the induction hypothesis or the choice of $a_{r+1, m+1}$. Therefore we must always have $i_{r+1} = j_t = m+1$ and so

$$\begin{aligned} &a_{r+2, i_{r+2}} \cdots a_{n, i_n} \left(a_{\sigma(t+1), j_{t+1}} \cdots a_{\sigma(n), j_n} \right)^{-1} \\ &\qquad = a^{-1}_{r+1, i_{r+1}} \left(a_{1, i_1} \cdots a_{r, i_r} \right)^{-1} a_{\sigma(1), j_1} \cdots a_{\sigma(t-1), j_{t-1}} a_{r+1, j_t}. \end{aligned}$$

Now we define g_σ and f_σ for all $\sigma \in S_n \backslash 1$ as

$$f_\sigma = \begin{cases} a_{r+2, i_{r+2}} \cdots a_{n, i_n} \left(a_{\sigma(t+1), j_{t+1}} \cdots a_{\sigma(n), j_n} \right)^{-1} & \text{if } 1 \le t \le n-1 \\ a_{r+2, i_{r+2}} \cdots a_{n, i_n} & \text{if } t = n \end{cases}$$

and

$$g_\sigma = \begin{cases} \left(a_{1, i_1} \cdots a_{r, i_r} \right)^{-1} a_{\sigma(1), j_1} \cdots a_{\sigma(t-1), j_{t-1}} & \text{if } 2 \le t \le n \\ \left(a_{1, i_1} \cdots a_{r, i_r} \right)^{-1} & \text{if } t = 1 \end{cases}$$

where $t = \sigma^{-1}(r+1)$. Hence $a_{r+1, m+1} \in S(g_\sigma, f_\sigma)$ and so

$$G = U_{r+1} \cup \left(\bigcup S(g_\sigma, f_\sigma) \right) \qquad (*)$$

where σ in $(*)$ varies over the set of all non-identity permutations in S_n such that $S(g_\sigma, f_\sigma) \neq \emptyset$. Obviously the set of pairs (g_σ, f_σ) is finite. Therefore $(*)$ shows that G is a finite union of right cosets of the centralizers of g_σ's. Thus by the famous theorem of Neumann [10] there exists g_σ in the FC-centre of G such that $S(g_\sigma, f_\sigma) \neq \emptyset$. But by the hypothesis $g_\sigma = f_\sigma = 1$. Thus there exist $(n-1)$-tuples $(i_1, \dots, i_r, i_{r+2}, \dots, i_n)$ and $(j_1, \dots, j_{t-1}, j_{t+1}, \dots, j_n)$ where $1 \leq i_1, \dots, i_r \leq m+1$, $1 \leq i_{r+2}, \dots, i_n \leq m$, $t = \sigma^{-1}(r+1)$, and $j_l = m+1$ whenever $1 \leq \sigma(l) \leq r$ and otherwise $j_l = m$ such that

$$a_{r+2, i_{r+2}} \cdots a_{n, i_n} \left(a_{\sigma(t+1), j_{t+1}} \cdots a_{\sigma(n), j_n} \right)^{-1}$$
$$= \left(a_{1, i_1} \cdots a_{r, i_r} \right)^{-1} a_{\sigma(1), j_1} \cdots a_{\sigma(t-1), j_{t-1}} = 1.$$

So for any $a \in X_{r+1, m}$ we have the following, which contradicts the induction hypothesis:

$$a_{1, i_1} \cdots a_{r, i_r} a a_{r+2, i_{r+2}} \cdots a_{n, i_n}$$
$$= a_{\sigma(1), j_1} a_{\sigma(2), j_2} \cdots a_{\sigma(t-1), j_{t-1}} a a_{\sigma(t+1), j_{t+1}} \cdots a_{\sigma(n), j_n}.$$

Therefore we have defined $X_{r+1, m+1}$. Thus we have inductively defined $X_{i, m} = \{a_{i,1}, \dots, a_{i,m}\}$ for all $m \in \mathbb{N}$ such that for all $\sigma \in S_n \backslash 1$

$$X_{1, m} \cdots X_{n, m} \cap X_{\sigma(1), m} \cdots X_{\sigma(n), m} = \emptyset.$$

Now set $X_i = \bigcup_{m=1}^{\infty} X_{i, m}$ $(i = 1, \dots, n)$, then X_i is infinite and

$$X_1 \cdots X_n \cap X_{\sigma(1)} \cdots X_{\sigma(n)} = \emptyset$$

for all $\sigma \in S_n \backslash 1$. Otherwise there exist n-tuples $(i_1, \dots, i_n)$ and $(j_1, \dots, j_n)$ on $\mathbb{N}$ and $\pi \in S_n \backslash 1$ such that $a_{1, i_1} \cdots a_{n, i_n} = a_{\pi(1), j_1} \cdots a_{\pi(n), j_n}$. Let $s = \operatorname{Max}\{i_1, \dots, i_n, j_1, \dots, j_n\}$. Then $X_{1, s} \cdots X_{n, s} \cap X_{\pi(1), s} \cdots X_{\pi(n), s} \neq \emptyset$, which is a contradiction with the construction of $X_{i, s}$ $(i = 1, \dots, n)$. ∎

By Lemma 2.3, every non-trivial restricted (∞, n)-permutable group has a non-trivial FC-element and since the class of restricted (∞, n)-permutable groups is closed under homomorphic images we have:

COROLLARY 2.4. *Every restricted (∞, n)-permutable group is FC-hypercentral.*

LEMMA 2.5. *Let G be an infinite restricted (∞, n)-permutable group. If G is finitely generated or non-periodic then G is an n-permutable group.*

Proof. Suppose that G is finitely generated. By Corollary 2.4, G is FC-hypercentral. Now by a result of McLain [7] (or see p. 133 of [11]) a finitely generated FC-hypercentral group is nilpotent-by-finite. Therefore G is a finitely generated nilpotent-by-finite group and so G is residually finite. Thus G is n-permutable by Lemma 2.1. Now assume that G is non-periodic. Then there is an element x of infinite order in G. Let $x_1, \ldots, x_n$ be arbitrary elements of G. By the previous part $\langle x, x_1, \ldots, x_n \rangle$ is an n-permutable group and so G is n-permutable. ∎

LEMMA 2.6. *Let G be a restricted (∞, n)-permutable group. Then G is hyperabelian-by-finite.*

Proof. We may assume that G is infinite, and it suffices to show that G contains a non-trivial normal abelian subgroup. Suppose no such normal abelian subgroup exists, and let x be a non-identity element in the FC-centre of G which exists by Lemma 2.3. Let $N_1 := \langle x \rangle^G$ be the normal closure of $\langle x \rangle$ in G, and let $C := C_G(N_1)$. Then $|G : C|$ is finite and $N_1 \cap C = Z(N_1)$ is a normal abelian subgroup of G. Hence $N_1 \cap C = 1$. Therefore N_1 is finite and, having a trivial centre, it is certainly non-abelian. Now suppose, inductively, that we have already defined normal non-abelian finite subgroups $N_1, \ldots, N_t$ of G such that $N_1, \ldots, N_t$ generate their direct product in G. Write $D := C_G(N_1 \cdots N_t)$; thus $|G : D|$ is finite. Now using Lemma 2.3 we can choose a non-trivial element y in the FC-centre of D. Then y is an element of the FC-centre of G. Let $N_{t+1} := \langle y \rangle^G$. It is easily seen that N_{t+1} is a finite non-abelian group. Moreover, $N_{t+1} \subset D$, so that $N_1, \ldots, N_t, N_{t+1}$ generate their direct product in G. Thus we have found in G an infinite direct product $N_1 \times N_2 \times \cdots \times N_t \times \cdots$ of finite non-abelian groups, which together with Lemma 2.2 gives a contradiction. ∎

LEMMA 2.7. *Let G be an infinite restricted (∞, n)-permutable group which is not Černikov. Then G is an n-permutable group.*

Proof. By Lemma 2.5, we may assume that G is periodic. By Lemma 2.6, there exists a normal hyperabelian subgroup H of finite index in G. Therefore H is a periodic locally soluble group and G is locally finite. Let $x_1, \ldots, x_n$ be arbitrary elements in G and let A be the finite subgroup generated by $x_1, \ldots, x_n$. We note that H is not a Černikov group and A can be regarded as a finite group of automorphisms of H. Now by a result of Zaicev [13], there exists an abelian subgroup B of H which is not Černikov and B is a normal subgroup of AB. Since B is periodic it is a direct product of the Sylow p-subgroups B_p of B. If infinitely many B_p are non-trivial, then since $|A|$ has only finitely many prime divisors, there exists an infinite subgroup D of B which is normal in AB such that $A \cap D = 1$. Consider the n infinite subsets $Dx_1, \ldots, Dx_n$. By the hypothesis there exists

$\sigma \in S_n \backslash 1$ such that

$$Dx_1 \cdots Dx_n \cap Dx_{\sigma(1)} \cdots Dx_{\sigma(n)} \neq \varnothing$$

and so $x_1 \cdots x_n (x_{\sigma(1)} \cdots x_{\sigma(n)})^{-1} \in A \cap D = 1$. Therefore $x_1 \cdots x_n = x_{\sigma(1)} \cdots x_{\sigma(n)}$ as required. So assume that there exist only finitely many B_p which are non-trivial. Since B is not a Černikov group and since the product of two normal Černikov subgroups of a group is a Černikov group, then there exists a prime number p such that B_p is not Černikov. Thus by Theorem 4.3.13 of [12], $C = \{b \in B \mid b^p = 1\}$ is an infinite elementary abelian p-group. Clearly C is normal in AB. Now the infinite group AC is a residually finite-by-finite group and so AC is residually finite. Therefore by Lemma 2.1, AC is an n-permutable group and the proof is complete. ∎

We need the following remark in the final step of the proof of the theorem. Here $|x|$ denotes the order of an element x of a group.

REMARK 2.8. We note that if $x_1, \ldots, x_n$ $(n > 1)$ are p-elements (p a prime) of distinct orders in an abelian group then $r < |x_1 \ldots x_n| \leq t$ where $r = \text{Min}\{|x_1|, \ldots, |x_n|\}$ and $t = \text{Max}\{|x_1|, \ldots, |x_n|\}$.

Proof of the Theorem. Let G be an infinite restricted (∞, n)-permutable group. By Lemma 2.7, we may assume that G is a Černikov group. Thus there exists an infinite normal subgroup A of G which is a direct product of finitely many groups isomorphic to C_{p^∞}, the quasicyclic p-group, for some prime number p. Let $x_1, \ldots, x_n \in G$ and let X be the finite subgroup generated by $x_1, \ldots, x_n$ (we note that G is locally finite). Let Y be the group of automorphisms of A induced by the elements of X under conjugation. Then Y is finite. Let α_0 be an integer such that $|a| \leq p^{\alpha_0}$ for any $a \in X \cap A$. By Lemma 3.5 of [4] there are infinite sequences $\alpha_0 < \alpha_1 < \cdots$ of integers and $a_1, a_2, \ldots$ of elements of A such that for any i, $|a_i| = p^{\alpha_i}$, and $|[a_i, y]| > p^{\alpha_{i-1}}$, for any $y \in Y \backslash C_Y(a_i)$. Now partition the set $\{a_i \mid i \geq 1\}$ into n infinite disjoint subsets J_i, $i = 1, \ldots, n$. Consider the set $J_i x_i$, $i = 1, \ldots, n$, and let $\sigma \in S_n \backslash 1$ be such that

$$(a_{i_1} x_1) \cdots (a_{i_n} x_n) = (a_{j_1} x_{\sigma(1)}) \cdots (a_{j_n} x_{\sigma(n)})$$

for suitable $a_{i_1} \in J_1, \ldots, a_{i_n} \in J_n$ and $a_{j_1} \in J_{\sigma(1)}, \ldots, a_{j_n} \in J_{\sigma(n)}$. Therefore

$$x := (x_1 \cdots x_n)^{-1} x_{\sigma(1)} \cdots x_{\sigma(n)} = a_{i_1}^{x_1 \cdots x_n} \cdots a_{i_n}^{x_n} \left(a_{j_1}^{x_{\sigma(1)} \cdots x_{\sigma(n)}} \ldots a_{j_n}^{x_{\sigma(n)}}\right)^{-1}.$$

We note that $i_1, \ldots, i_n$ are pairwise distinct as are $j_1, \ldots, j_n$. If

$$\{i_1, \ldots, i_n\} \cap \{j_1, \ldots, j_n\} = \varnothing$$

then by Remark 2.8, $p^{\alpha_r} < |x|$ where $r = \mathrm{Min}\{i_1, \ldots, i_n, j_1, \ldots, j_n\}$, which is a contradiction, since $\alpha_r > \alpha_0$ and $x \in X \cap A$. Thus $F = \{i_1, \ldots, i_n\} \cap \{j_1, \ldots, j_n\} \neq \emptyset$. Let $|F| = s$. We may assume, without loss of generality, that $i_1 = j_1, \ldots, i_s = j_s$. Then we may write

$$x = [a_{i_1}, y_1]^{z_1} \cdots [a_{i_s}, y_s]^{z_s} a_{i_{s+1}}^{y_{s+1}} \cdots a_{i_n}^{y_n} \left(a_{j_{s+1}}^{z_{s+1}} \cdots a_{j_n}^{z_n}\right)^{-1},$$

for some $y_1, \ldots, y_n, z_1, \ldots, z_n \in X$. Now suppose, for a contradiction, that $x \neq 1$. If $[a_{i_1}, y_1] = \cdots = [a_{i_s}, y_s] = 1$ then $s < n$, since $x \neq 1$. Then since $i_{s+1}, \ldots, i_n, j_{s+1}, \ldots, j_n$ are pairwise distinct, by Remark 2.8, $|x| > p^{\alpha_k}$ where $k = \mathrm{Min}\{i_{s+1}, \ldots, i_n, j_{s+1}, \ldots, j_n\}$, which is a contradiction. Thus we may assume, without loss of generality, that $y_l \in Y \backslash C_Y(a_{i_l})$, for $l = 1, \ldots, s$ and $i_1 < \cdots < i_s$. Now we claim that the elements

$$[a_{i_1}, y_1], \ldots, [a_{i_s}, y_s], a_{i_{s+1}}^{y_{s+1}}, \ldots, a_{i_n}^{y_n}, a_{j_{s+1}}^{z_{s+1}}, \ldots, a_{j_n}^{z_n}$$

have distinct orders. For, since $p^{\alpha_{i_l - 1}} < |[a_{i_l}, y_l]| \leq p^{\alpha_{i_l}}$ for $l = 1, \ldots, s$ and $\alpha_{i_1} < \cdots < \alpha_{i_s}$, then the elements $[a_{i_1}, y_1], \ldots, [a_{i_s}, y_s]$ have distinct orders. Clearly $a_{i_{s+1}}^{y_{s+1}}, \ldots, a_{i_n}^{y_n}$, $a_{j_{s+1}}^{z_{s+1}}, \ldots, a_{j_n}^{z_n}$ have distinct orders. If there exist $l \in \{1, \ldots, s\}$ and $k \in \{s+1, \ldots, n\}$ such that $|[a_{i_l}, y_l]| = |a_{i_k}|$ or $|[a_{i_l}, y_l]| = |a_{j_k}|$ then since $p^{\alpha_{i_l - 1}} < |[a_{i_l}, y_l]| \leq p^{\alpha_{i_l}}$, $\alpha_{i_l} = \alpha_{i_k}$ or $\alpha_{i_l} = \alpha_{j_k}$ and so $i_l = i_k$ or $i_l = j_k$, a contradiction. Now by Remark 2.8, $p^t < |x|$, where

$$p^t = \mathrm{Min}\{|[a_{i_1}, y_1]|, \ldots, |[a_{i_s}, y_s]|, |a_{i_{s+1}}^{y_{s+1}}|, \ldots, |a_{i_n}^{y_n}|, |a_{j_{s+1}}^{z_{s+1}}|, \ldots, |a_{j_n}^{z_n}|\}.$$

But $t > \alpha_0$ which is a contradiction. ∎

REFERENCES

1. R. D. Blyth, Rewriting products of group elements, I, *J. Algebra* **116** (1988), 506–521.
2. R. D. Blyth, Rewriting products of group elements, II, *J. Algebra* **119** (1988), 246–259.
3. M. Curzio, P. Longobardi, M. Maj, and D. J. S. Robinson, A permutational property of groups, *Arch. Math.* **44** (1985), 383–389.
4. P. Longobardi, M. Maj, and A. H. Rhemtulla, Infinite groups in a given variety and Ramsey's theorem, *Comm. Algebra* **20** (1992), 127–139.
5. P. Longobardi, M. Maj, and S. E. Stonehewer, The classification of groups in which every product of four elements can be reordered, *Rend. Sem. Mat. Univ. Padova* **93** (1995), 7–26.
6. M. Maj, On the derived length of groups with some permutational property, *J. Algebra* **136** (1991), 86–91.
7. D. H. McLain, Remarks on the upper central series of a group, *Proc. Glasgow Math. Assoc.* **3** (1956), 38–44.
8. A. Mohammadi Hassanabadi, A property equivalent to permutability for groups, *Rend. Sem. Mat. Univ. Padova*, **100** (1998), 137–142.
9. A. Mohammadi Hassanabadi and A. H. Rhemtulla, Criteria for commutativity in large groups, *Canad. Math. Bull.* **41**, No. 1 (1998), 65–70.

10. B. H. Neumann, Groups covered by permutable subsets, *J. London Math. Soc.* **29** (1954), 236–248.
11. D. J. S. Robinson, "Finiteness Conditions and Generalized Soluble Groups, Part 1," Springer-Verlag, Berlin, 1972.
12. D. J. S. Robinson, "A Course in the Theory of Groups," 2nd ed., Springer-Verlag, Berlin, 1996.
13. D. I. Zaicev, On solvable subgroups of locally solvable groups, *Dokl. Akad. Nauk SSSR* **214** (1974), 1250–1253. [Translation in *Soviet Math. Dokl.* **15** (1974), 342–345.]